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Correlation functions of quantum q-oscillators

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Abstract

Nonlinearity of electromagnetic field vibrations described by q-oscillators is shown to produce essential dependence of second correlation functions on intensity and deformation of Planck distribution. Experimental tests of such nonlinearity are suggested.

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In the framework of investigations on quantum groups [1] the special case of the Heisenberg-Weyl algebra was first considered by Biedenharn [2] and Macfarlane [3]. They introduced "q-oscillator operators" starting from a deformed commutation relation containing a dimensionless parameter q. The physical sense of the quantun q-oscillator up to now was not clarified. In [4] a generalization to several modes has been analyzed. An attempt to connect quantum q-oscillator with relativistic oscillator model has been discussed in [5]. The q-oscillators and quantum group $SU_q(2)$ have also been considered in the frame of generalized James-Cummings model [6]. Nevertheless these attempts are based on purely mathematical properties of these operators. In [7] aspects of the physical nature of q-oscillator have been considered by the present authors. It was seen that it is subject to nonlinear vibrations with a special kind of dependence of the frequency on the amplitude and the influence of such nonlinearity on Bose distribution function was evaluated. Very recentlya deformation of Bose distribution function has been studied also in [8]. The aim of this letter is to study the influence of the nonlinearity on the correlation function of q-oscillator in different states, namely coherent and squeezed ones and in a thermal bath. We will show that such quantities acquire a dependence on the momenta of the number operator. Assuming that the electromagnetic field be described in terms of q-oscillators we will discuss the physical consequences of this suggestion, namely how the presence of the deformation parameter q changes the second order correlation formulae. In [7] it was shown that the classical q-oscillator of amplitude $|\alpha|$ vibrates with a frequency

$$\omega_q = \omega \frac{\kappa \cosh \kappa |\alpha|^2}{\sinh \kappa} \tag{1}$$

with $\omega \in \mathbf{R}$ and $\kappa = logq$. For $q \to 1$ linearity of vibration is recovered. q-oscillator operators can be defined anyhow also in terms of the usual harmonic oscillator variables $(a, a^\dagger)$

$$a_q = a\sqrt{\frac{\sinh \hat{n}\kappa}{\hat{n}\sinh \kappa}} \tag{2}$$

$$a_q^\dagger = \sqrt{\frac{\sinh \hat{n}\kappa}{\hat{n}\sinh \kappa}}a^\dagger \tag{3}$$

in which $\hat{n} = a^\dagger a$. The quantum system under consideration then continues to be described by such last variables $(a, a^\dagger)$, but it evolves under a Hamiltonian which in dimensionless units ($\omega = 1$) reads

$$H_q(\hat{n}) = \frac{1}{2}\frac{1}{\sinh\kappa}(\sinh\kappa(\hat{n}+1) + \sinh\kappa\hat{n}). \tag{4}$$

In terms of $(a_q, a_q^\dagger)$ it reads

$$H_q(\hat{n}) = \frac{1}{2}(a_q a_q^\dagger + a_q^\dagger a_q). \tag{5}$$

It is easy to prove that for an arbitrary function $V(\hat{n})$ which can be represented as a series of powers of $\hat{n}$ the following string of relations holds

$$V(\hat{n})a = aV(\hat{n}-1), \quad V(\hat{n})a^\dagger = a^\dagger V(\hat{n}+1). \tag{6}$$

The definition of second correlation function we will use is

$$\gamma(t) = Tr\rho a^\dagger(t)a(0) \tag{7}$$

with ρ standing for the density matrix. Other definitions exist for such a quantity, for instance with a normalization factor, but we take the above which is simpler, as we wish to give an idea of the corrections due to q-deformation, namely due to evolution under H_q. We have therefore, using the string relations,

$$\gamma(t) = Tr\rho e^{itH_q(\hat{n})}a^\dagger(0)e^{-itH_q(\hat{n})}a(0) = Tr\rho e^{it(H_q(\hat{n})-H_q(\hat{n}-1))}\hat{n}. \tag{8}$$

We will suppose nonlinearity appearing only at high intensities, so that the limiting process $q \to 1(\kappa \to 0)$ is physically relevant. We have

$$H_q(\hat{n}) - H_q(\hat{n}-1) = \frac{1}{2\sinh\kappa}(\sinh\kappa(\hat{n}+1) + \sinh\kappa(\hat{n}-1)) \simeq 1 + \frac{\kappa^2\hat{n}^2}{2}. \tag{9}$$

In what follows we will take all quantities in such an approximation so that

$$\gamma(t) \simeq e^{it}Tr\rho(1 + it\frac{\kappa^2\hat{n}^2}{2})\hat{n}. \tag{10}$$

This formula is correct for finite times, not considering yet the asymptotic behaviour of (8) for large times. We consider first the case of a coherent state $|\alpha>, \alpha \in \mathbf{C}$, for which the density matrix is

$$\rho^{(coh)} = e^{|\alpha|^2}\sum_{m,n}\frac{\alpha^n\alpha^{*m}}{\sqrt{n!m!}}|n><m| \tag{11}$$

and therefore

$$\gamma^{(coh)}(t) \simeq |\alpha|^2 e^{it}(1 + \frac{it\kappa^2}{2}(|\alpha|^4 + 3|\alpha|^2 + 1)). \tag{12}$$

We next consider the q-oscillator in the squeezed vacuum state

$$|0_s >= e^{\frac{r}{2}(a^2 - a^{\dagger 2})}|0 >, \tag{13}$$

where r is the squeezing parameter defined by the equations

$$< 0_s|\frac{(a + a^\dagger)^2}{2}|0_s > \;=\; \frac{1}{2}e^{-r} \tag{14}$$

$$< 0_s|\frac{(a - a^\dagger)^2}{2}|0_s > \;=\; -\frac{1}{2}e^{r}. \tag{15}$$

The density matrix is now

$$\rho^{(sq)} = |0_s >< 0_s| \tag{16}$$

and the second order correlation function in the above approximation is

$$\gamma^{(sq)}(t) \simeq e^{it}(n_s + \frac{it\kappa^2}{2}n_s^3), \tag{17}$$

where

$$n_s =< 0_s|\hat{n}|0_s >= \sinh^2 r \tag{18}$$

and

$$n_s^3 =< 0_s|\hat{n}^3|0_s >= \sinh^2 2r \cosh 2r + \frac{3}{2}\sinh^2 2r \sinh^2 r + \sinh^6 r. \tag{19}$$

The density matrix for the thermal case is

$$\rho^{(th)} = e^{\frac{-\beta}{2}(a_q^\dagger a_q + a_q a_q^\dagger)}\frac{1}{Z_q} \tag{20}$$

with

$$Z_q = \sum_{n=0}^{\infty} \exp\left[-\frac{\beta}{2}\frac{\sinh\kappa(n+1) + \sinh\kappa n}{\sinh\kappa}\right] \tag{21}$$

and the parameter $\beta = \hbar\omega/kT$, with $\hbar$ Planck constant, ω frequency, k Boltzmann constant and T absolute temperature. In the considered approximation we write

$$Z_q \simeq Z + a\kappa^2 \tag{22}$$

where Z is the partition function for the harmonic oscillator

$$Z = \frac{1}{2\sinh\frac{\beta}{2}} \tag{23}$$

and the coefficient a can be calculated [7] and has the value

$$a = -\frac{\beta Z}{12}(2\overline{n^3} + 3\overline{n^2} + \bar{n}). \tag{24}$$

Then

$$\rho^{(th)} \simeq \frac{e^{-\beta(\hat{n}+\frac{1}{2})}}{Z}(1 - \kappa^2(\frac{a}{Z} + \frac{\beta}{12}(2\hat{n}^3 + 3\hat{n}^2 + \hat{n}))) \tag{25}$$

and, inserting this in (10), we obtain

$$\gamma^{(th)}(t) \simeq e^{it}\{\bar{n} + \kappa^2[\frac{it}{2}\overline{n^3} + \frac{\beta}{6}\overline{n^3}\bar{n} + \frac{\beta}{4}\overline{n^2}\bar{n} + \frac{\beta}{12}\bar{n}^2 - \frac{\beta}{6}\overline{n^4} - \frac{\beta}{4}\overline{n^3} - \frac{\beta}{12}\overline{n^2}]\}. \tag{26}$$

All the above equations contain some momenta of $\hat{n}$ in Planck distribution

$$\bar{n} = \frac{1}{e^\beta - 1}, \tag{27}$$

$$\overline{n^2} = \frac{e^\beta + 1}{(e^\beta - 1)^2}. \tag{28}$$

$$\overline{n^3} = \frac{4e^\beta + e^{2\beta} + 1}{(e^\beta - 1)^3}, \tag{29}$$

$$\overline{n^4} = \frac{e^{3\beta} + 11e^{2\beta} + 11e^\beta + 1}{(e^\beta - 1)^4}. \tag{30}$$

For $t = 0$ the second correlation function gives the mean value of number of photons $(\bar{n})_q$ and so for the thermal state we have the deformed Planck distribution formula [7]

$$(\bar{n})_q \simeq \frac{1}{e^{\frac{\hbar\omega}{kT}} - 1} - \kappa^2\frac{\hbar\omega}{kT}\frac{e^{\frac{3\hbar\omega}{kT}} + 4e^{\frac{2\hbar\omega}{kT}} + e^{\frac{\hbar\omega}{kT}}}{(e^{\frac{\hbar\omega}{kT}} - 1)^4}. \tag{31}$$

Here the first term is the usual Planck distribution formula and its correction is proportional to the square of the nonlinearity parameter κ.

As it was seen the classical one-dimensional q-oscillator is nothing else than a nonlinear oscillator with a very specific type of the nonlinearity. Namely, its frequency depends on its energy as hyperbolic cosine of the energy. As a consequence spectra should show a blue-shift effect when intensity of photon beams increases. Should this phenomenon be observed it would be possibly explained also by other types of nonlinearity.

The second-order correlation functions studied by us depend on the higher momenta of the Bose distribution for the photon number operator. For squeezed states, it is remarkable the larger dependence of the correlation function on the squeezing parameter, comparing with the standard oscillator. As it was seen, the discussed q-nonlinearity produces a correction to Planck distribution formula and also this may be subjected to an experimental test. All these properties mark the main difference with the standard, linear electromagnetism.

Therefore, in spite of the mathematical beauty of q-deformation procedure, the most important issue is to envisage how to test experimentally the possibility of describing electrodynamic phenomena by means of these q-oscillators. At a first thought , non-linear optics experiments appear as potential candidates for obtaining upper bounds for the value of κ. Another class of experiments could be based on the dependence of time-correlation function on the intensity. The interpretation of these experiments does not depend critically on the development of a q-field QED. In fact, these measurements can be carried out in vacuum. For example, a continuous laser beam could be split in two beams of different intensities and sent to two spectra analyzers. Measuring the widths of the two spectra it would be possible to put an upper limit to the deformation parameter κ. These measurements could last for years thus guaranteeing a signal-to-noise ratio adequate for appreciating very small values of the parameter κ.

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